Stoer-Wagner Algorithm

A Minimum Cut Algorithm for Undirected Graphs

BigNews

CS214: Algorithms and Complexity
Shanghai Jiao Tong University

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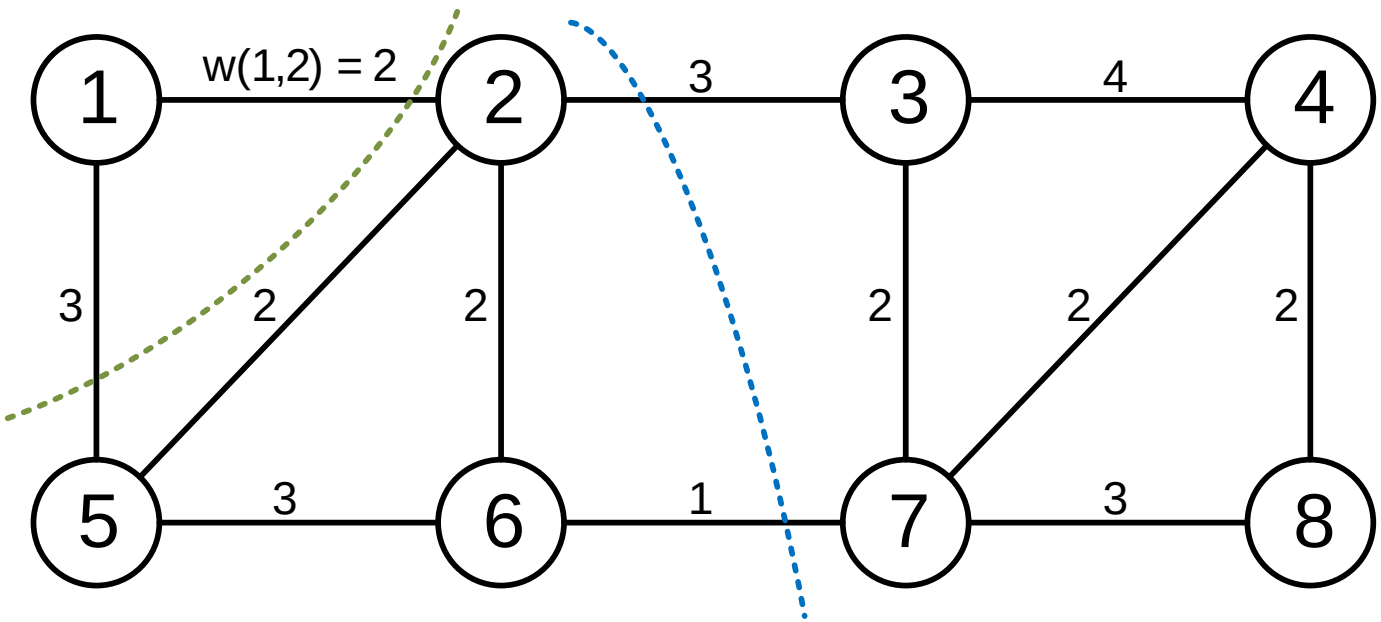
Outline

Global Minimum Cut

DeÞnition

Given an undirected graph $G = (V, E)$, a global min-cut is a partition of V into two subsets (S, T) such that the sum of weights of edges between S and T is minimized.



Outline

Stoer-Wagner Algorithm (1)

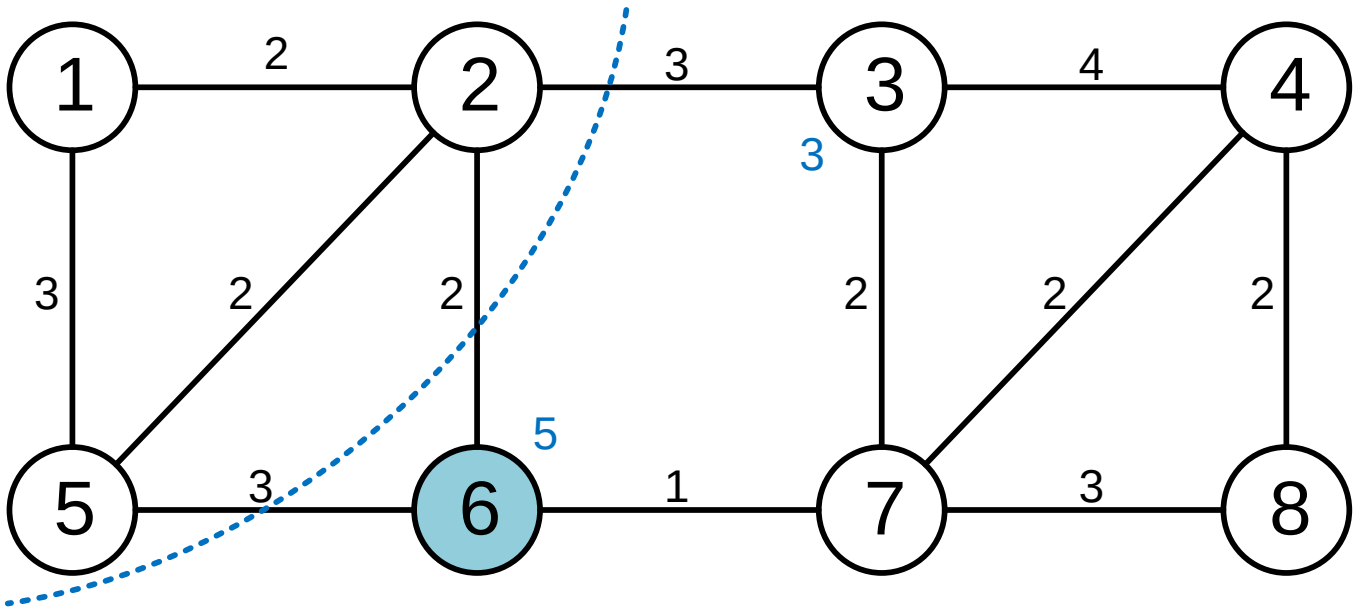
MinimumCutPhase(G, w, a)

```
1 A ! {  a}
2 while (A ⫦ V) do
3 |   add toA the most tightly connected vertex
4 end
5 store thœcut-of-the-phase
6 shrinkG by merging the two vertices added last
```

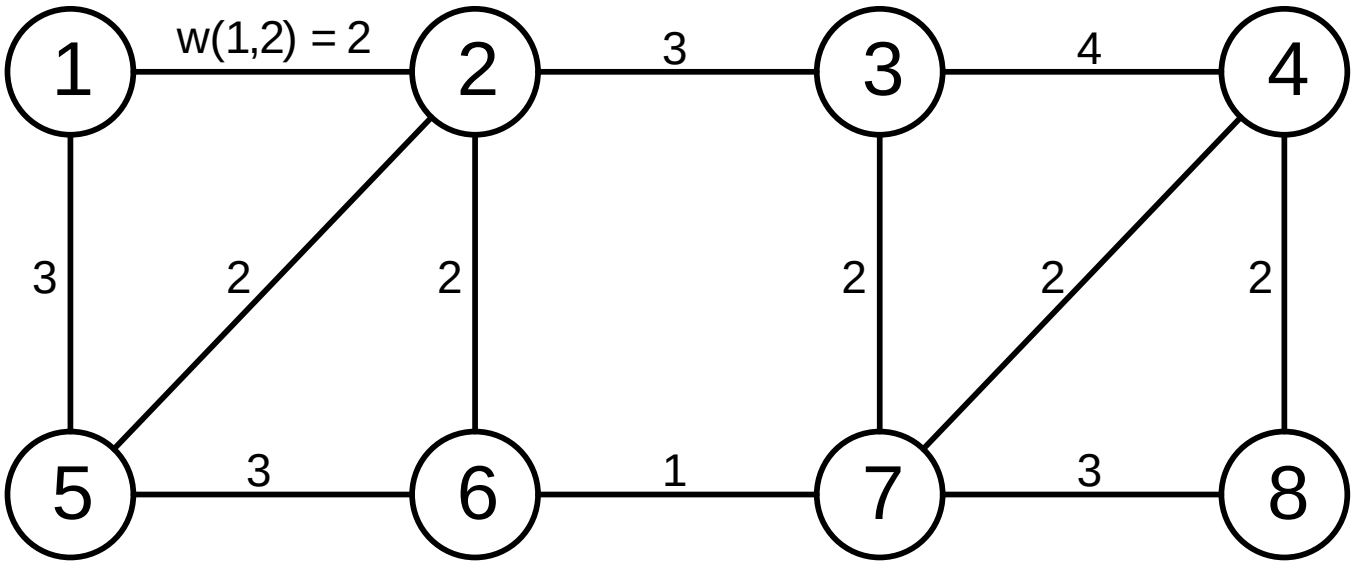
Most Tightly Connected

Definition

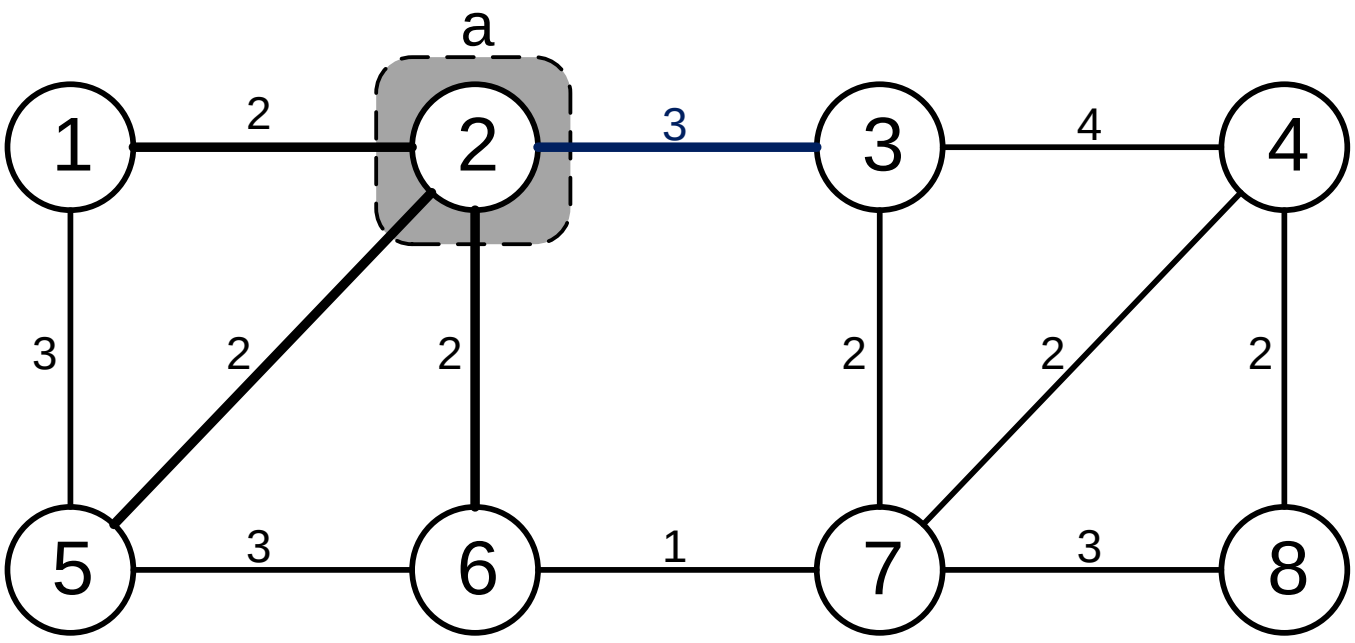
A vertex t are called most tightly connected to vertex set A if sum of the weights of the edges connected between t and A is the highest among other vertices not belonging to A.



A graphG = (V, E) with edge-weights

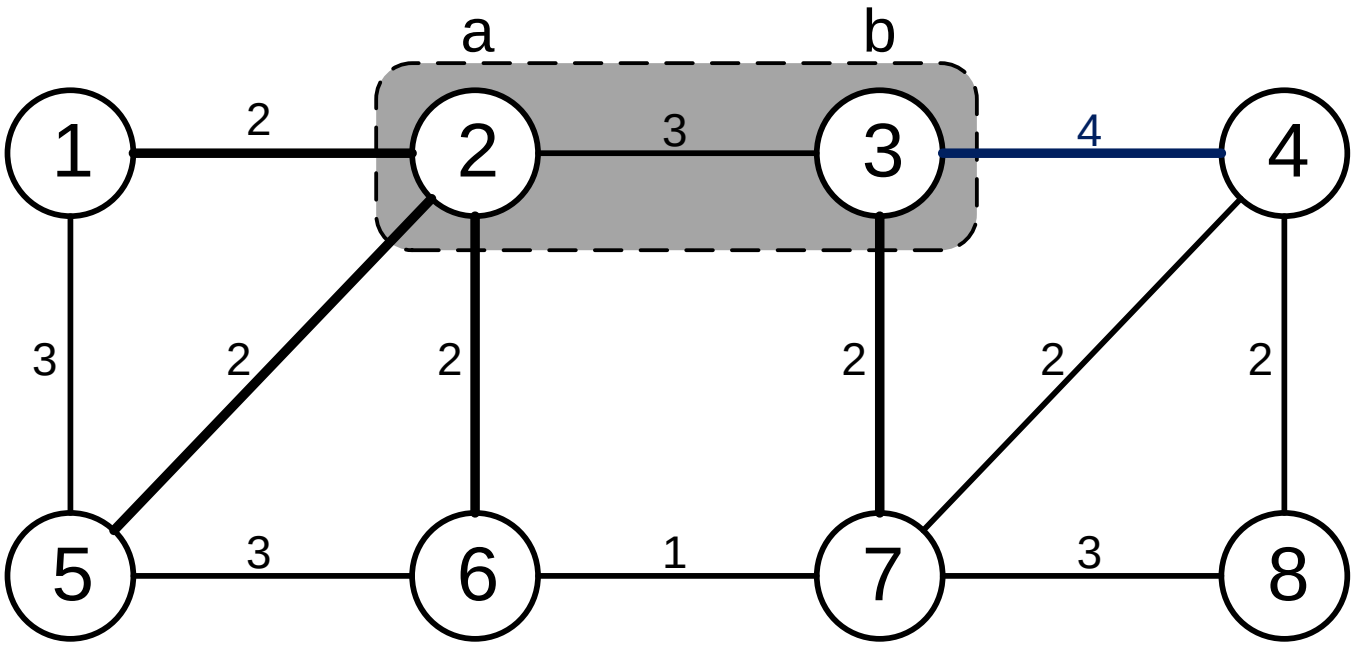


Start with vertex a = 2



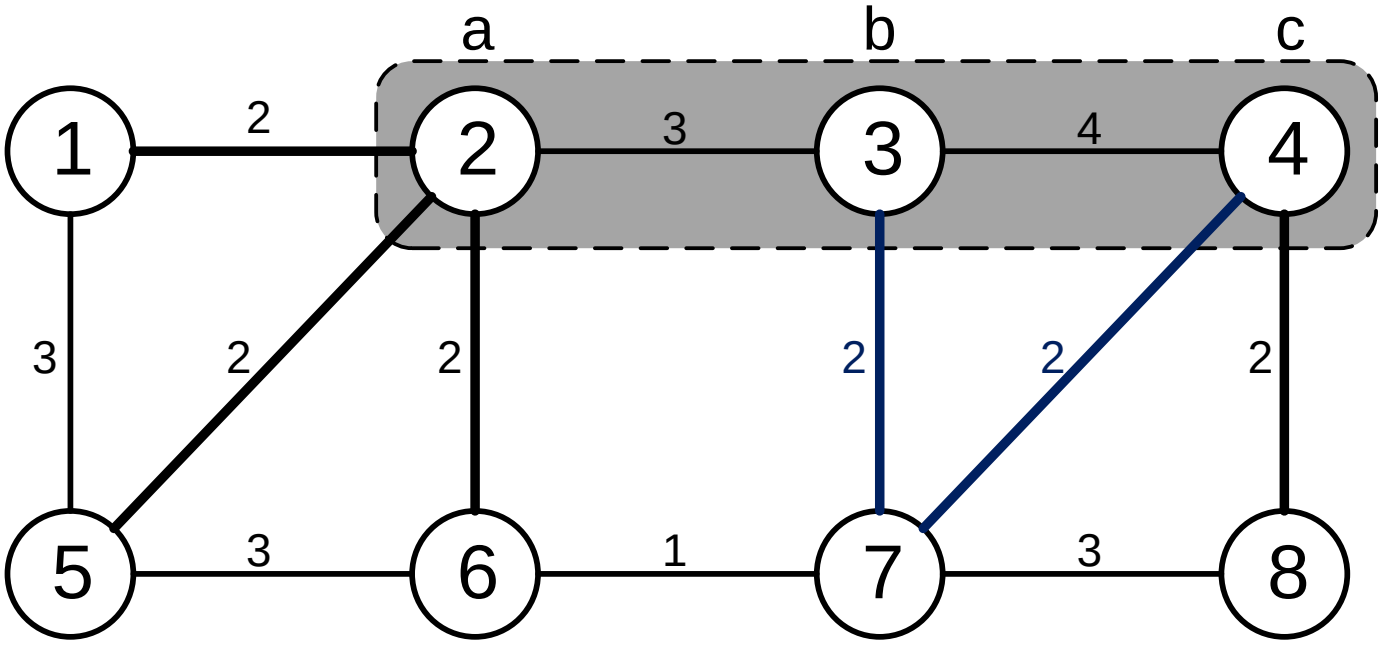
A = {2}

Add vertex b = 3



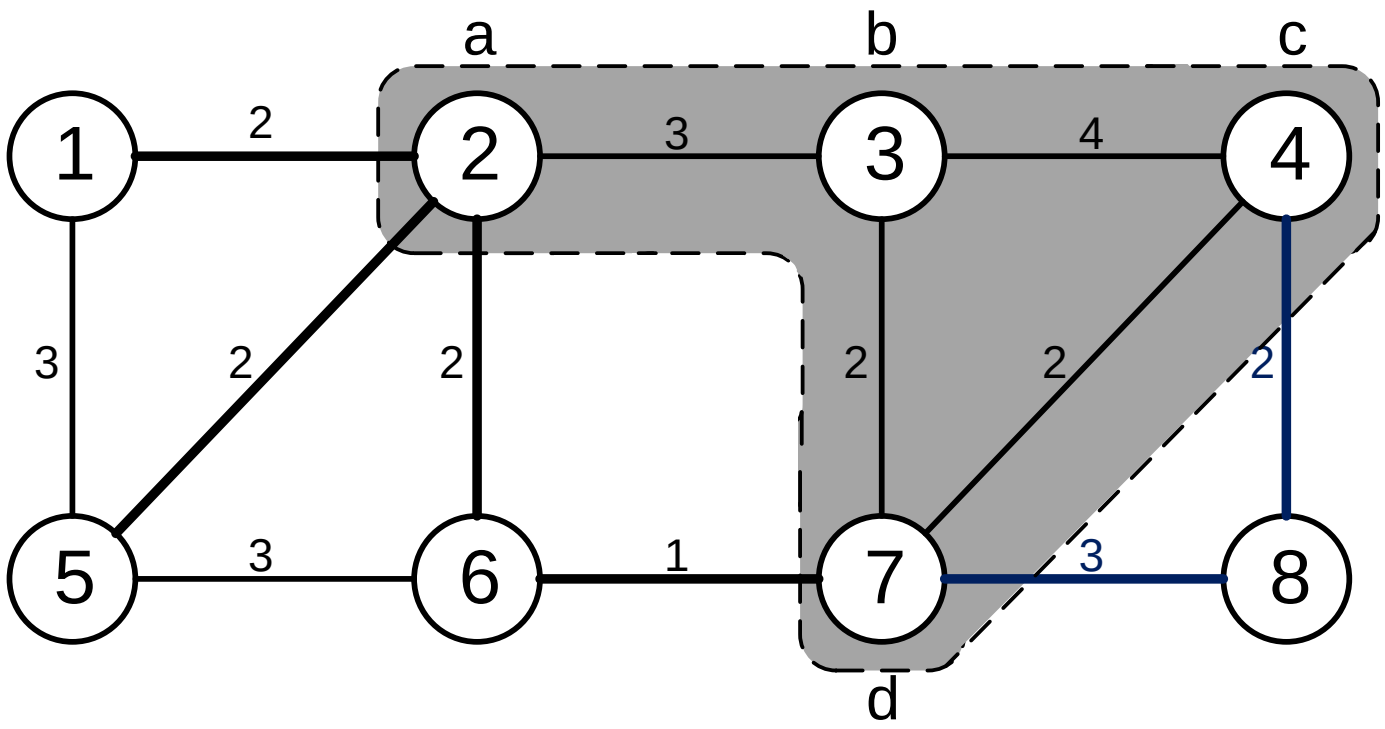
A = {2, 3}

Add vertex c = 4



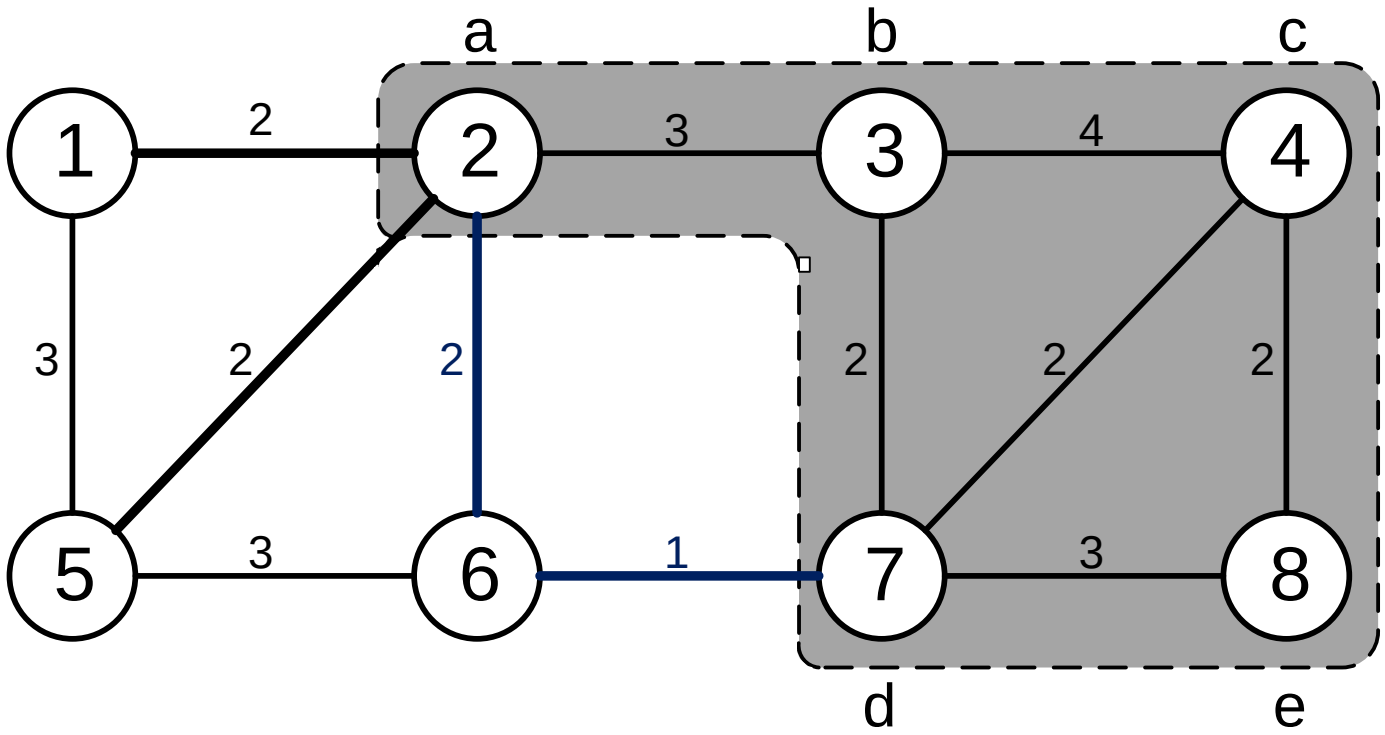
A = $\{2, 3, 4\}$

Add vertex d = 7



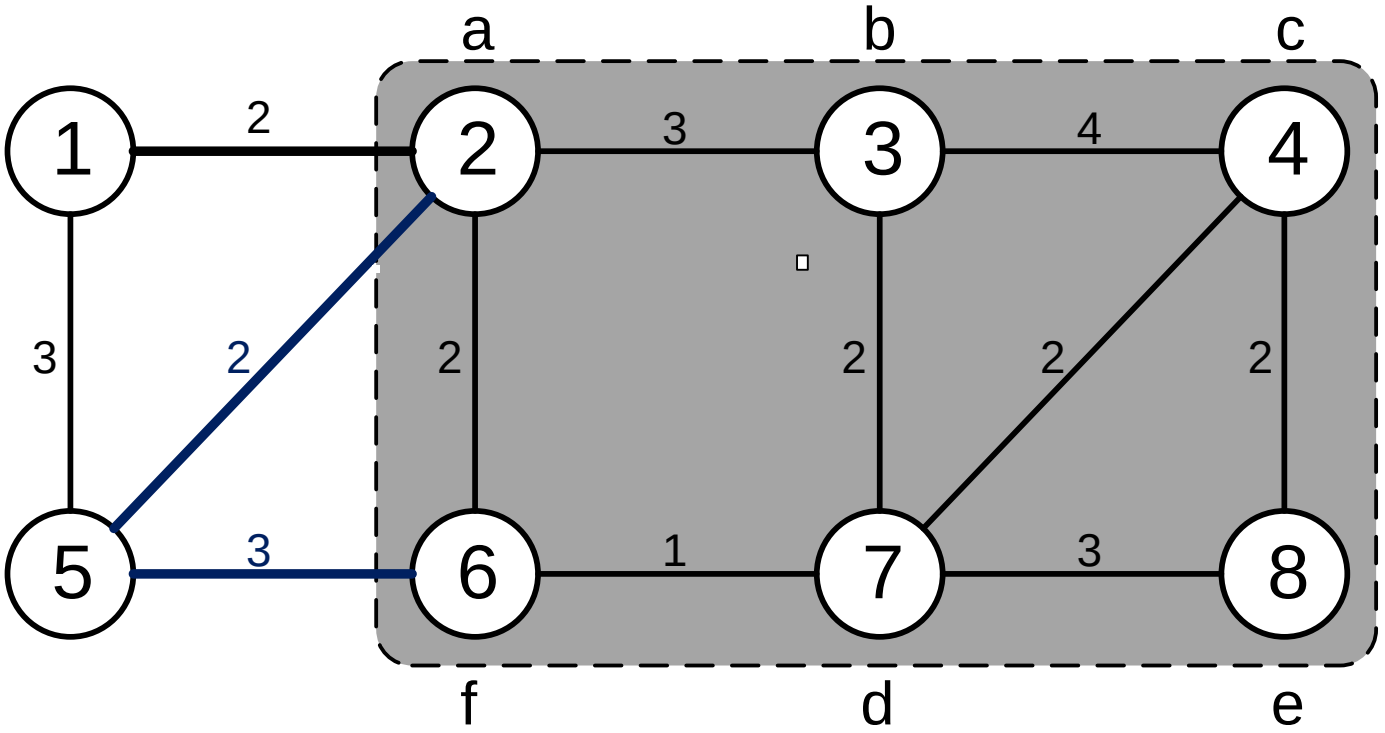
A = $\{2, 3, 4, 7\}$

Add vertex e = 8



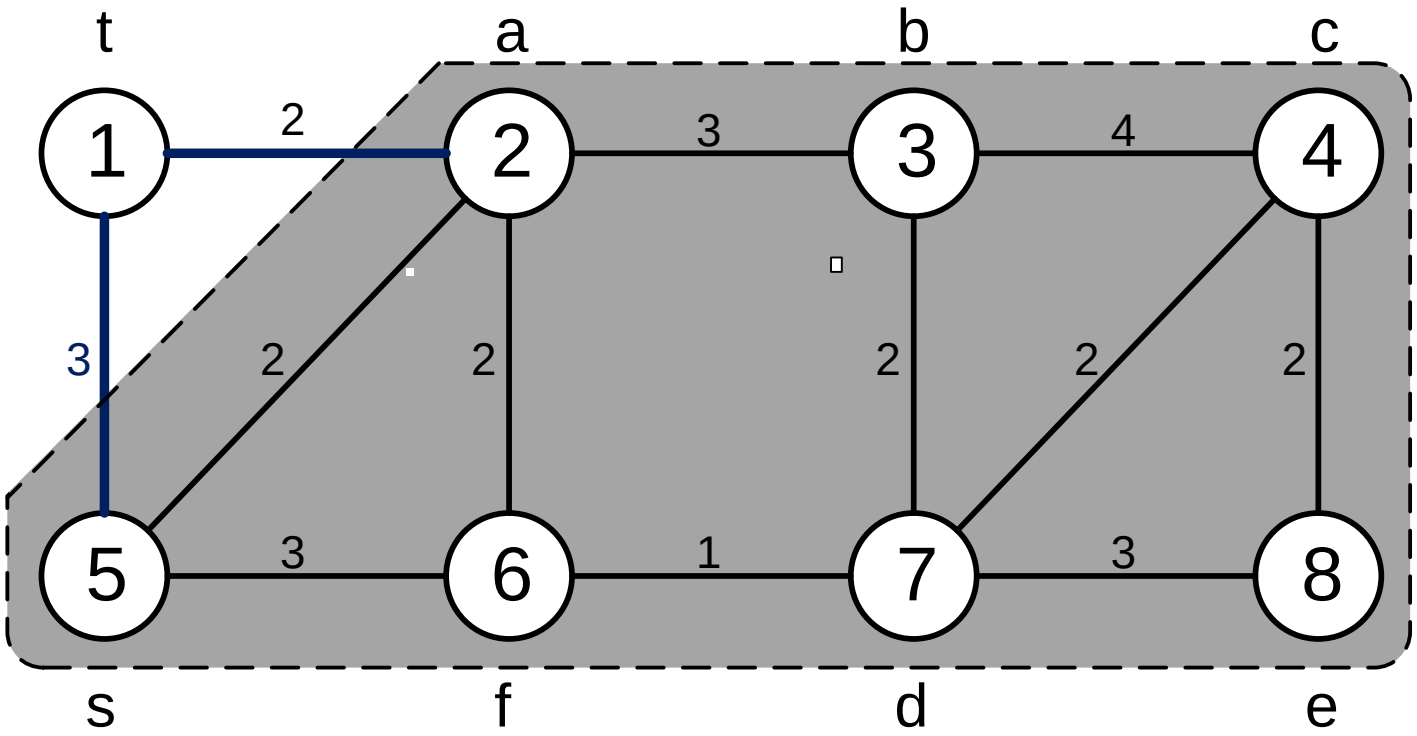
A = $\{2, 3, 4, 7, 8\}$

Add vertex f = 6



A = {2, 3, 4, 7, 8, 6}

Add vertex s = 5 and vertex t = 1



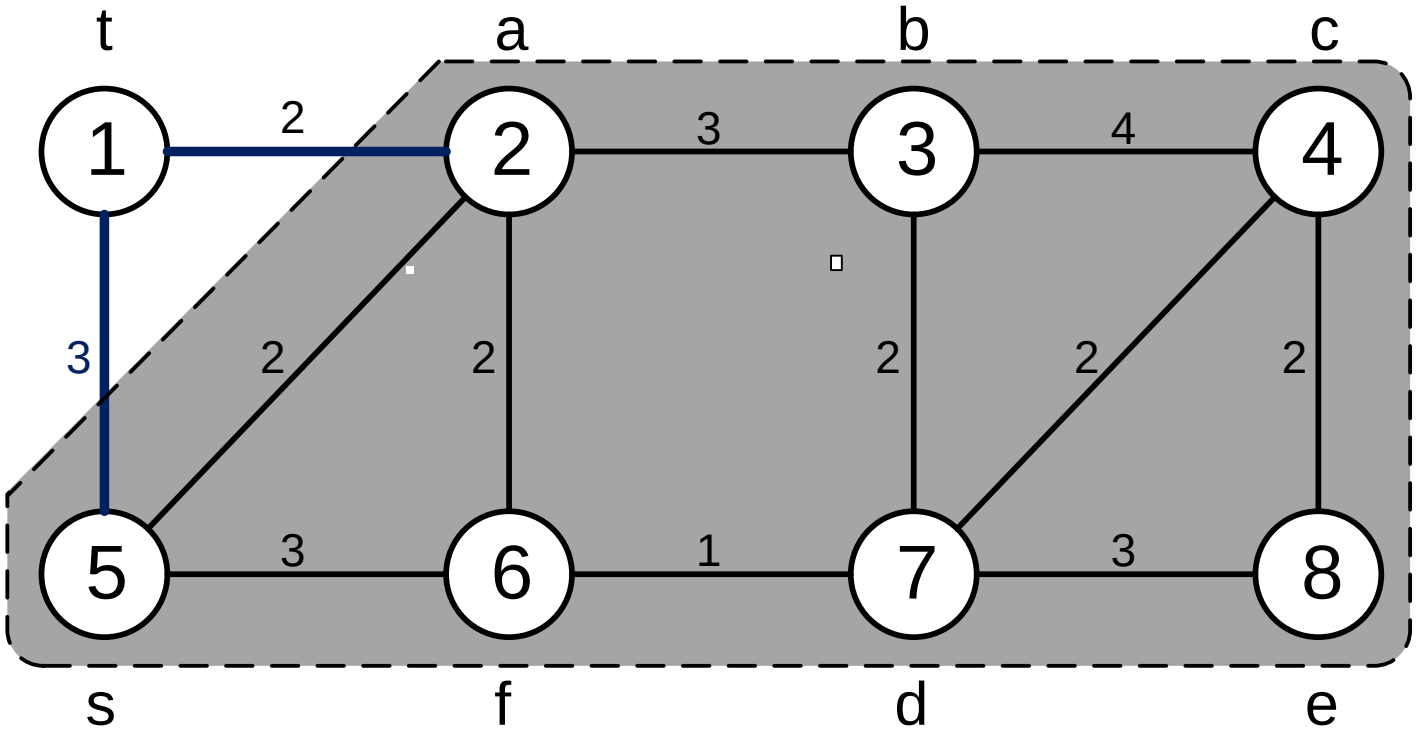
A = {2, 3, 4, 7, 8, 6, 5}

A = {2, 3, 4, 7, 8, 6, 5, 1}

s and t are the last two vertices (in order) added to A, and we get a cut C(A-t, t), which is so-called cut-of-the-phase.

Lemma

Each cut-of-the-phase is a minimum s-t cut in the current graph, where s and t are the two vertices added last in the phase.



Implications

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Then min s-t cut is also a global min cut of G.

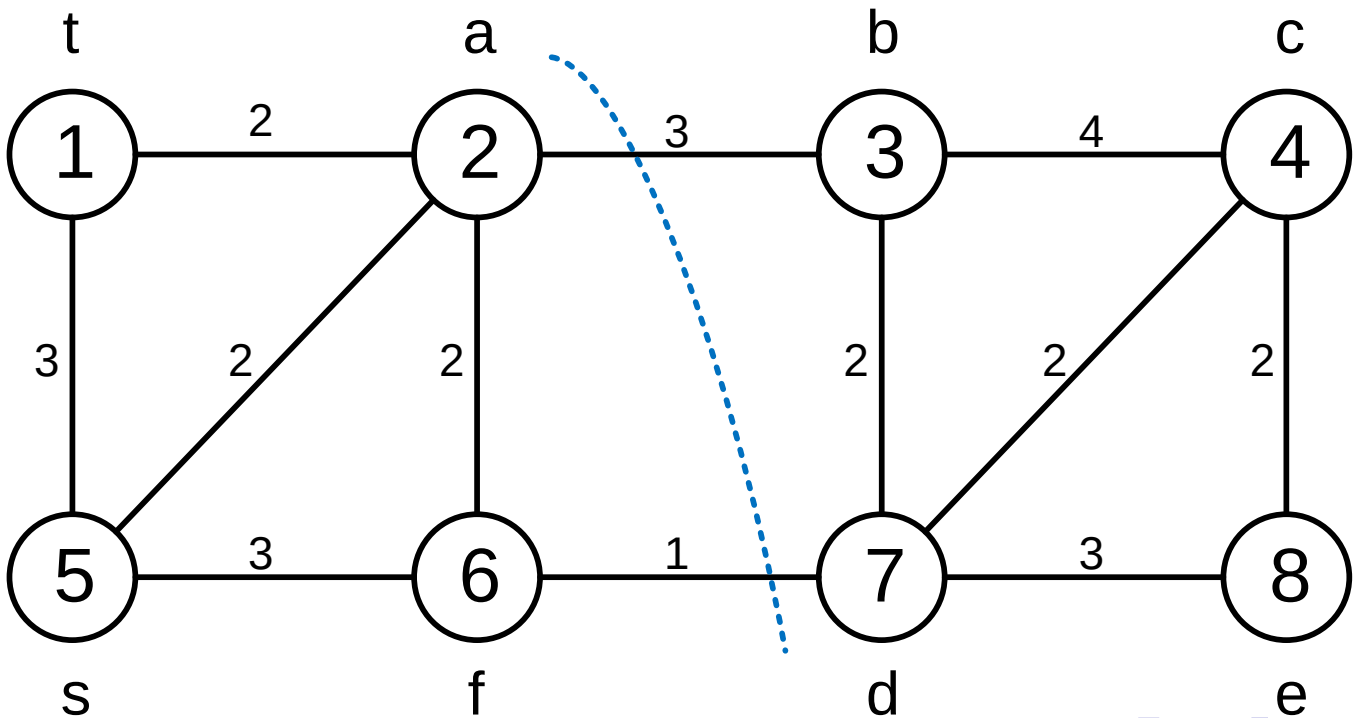
What if min cut of G does not separate s and t?

Then s and t are in the same partition of the global min cut, and we can merge them without changing the global min cut.

Merge

DeÞnition

The two vertices are replaced by a new vertex and any edges from the two vertices to a remaining vertex are replaced by an edge weighted by the sum of the weights of the previous two edges, while edges joining the merged nodes are removed.



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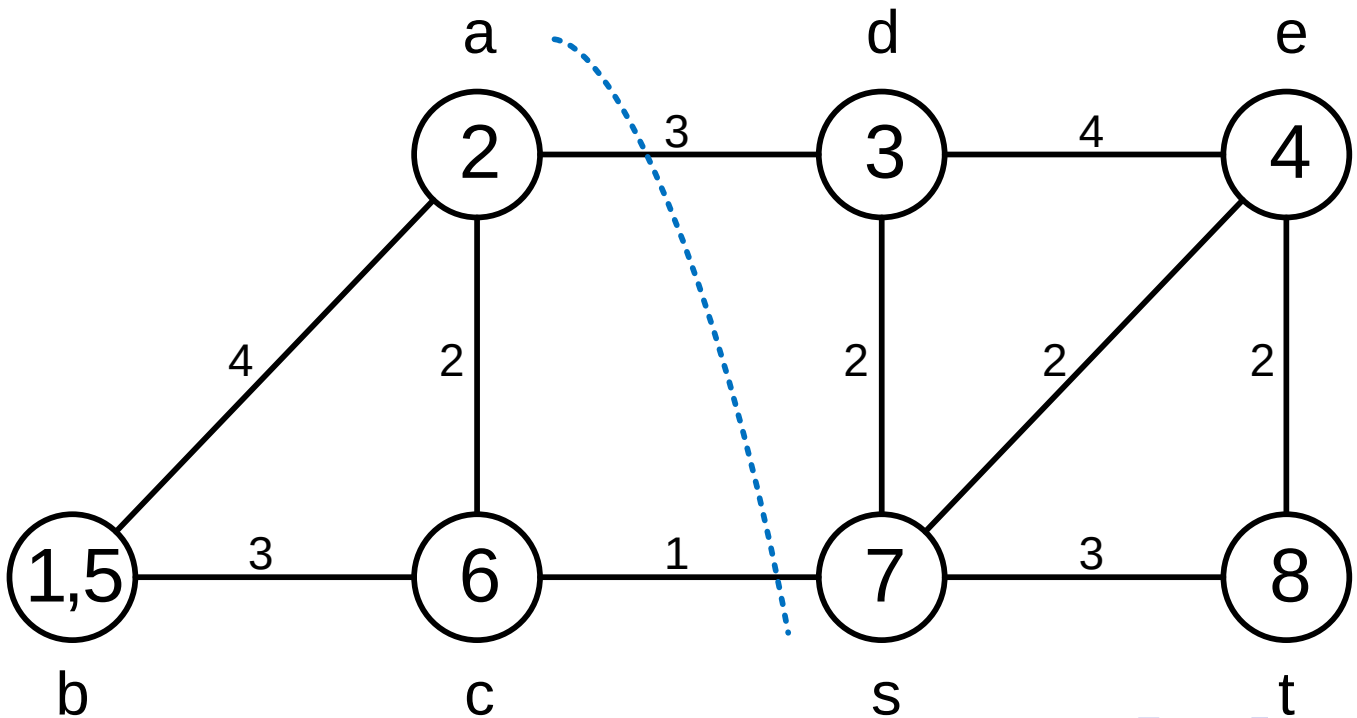
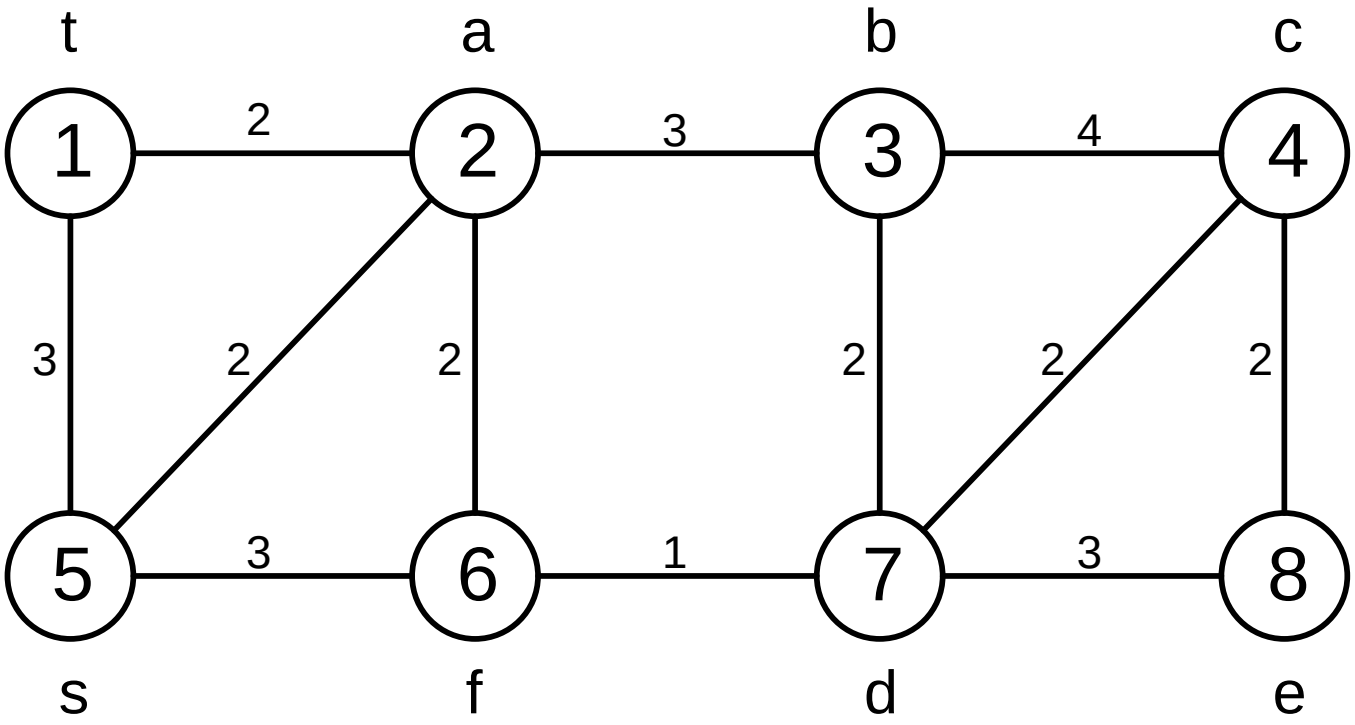


Stoer-Wagner Algorithm (2)

```
MinimumCut (G, w, a)
1 while (| V |> 1) do
2 |   MinimumCutPhase(G, w, a)
3 |   if the cut-of-the-phase is lighter than the current minimum cut
  |   then
4 |   |   store the cut-of-the-phase as the current minimum cut
5 |   end
6 end
```

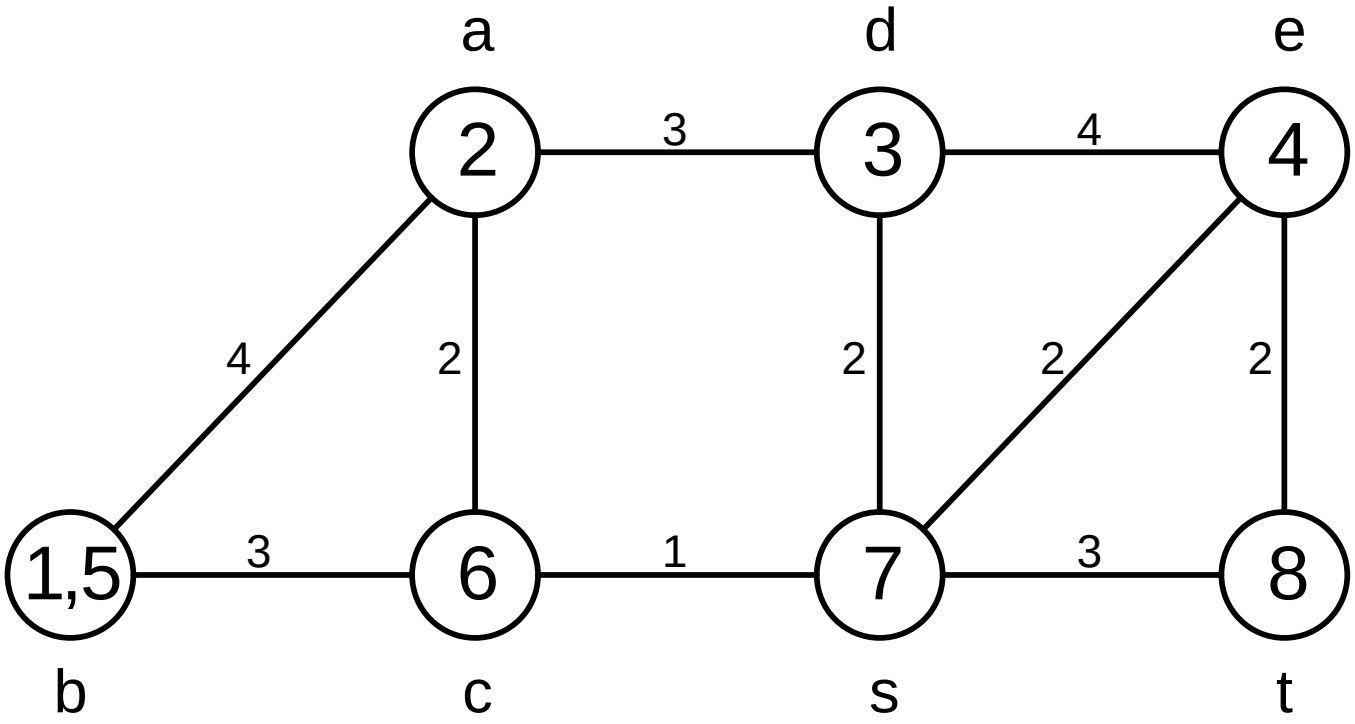
After the 1st MinimumCutPhase(G, !, a), a = 2



vertex ordering: a, b, c, d, e, f , s, t

cut-of-the-phase: { 1} , { 2, 3, 4, 5, 6, 7, 8} ! = 5

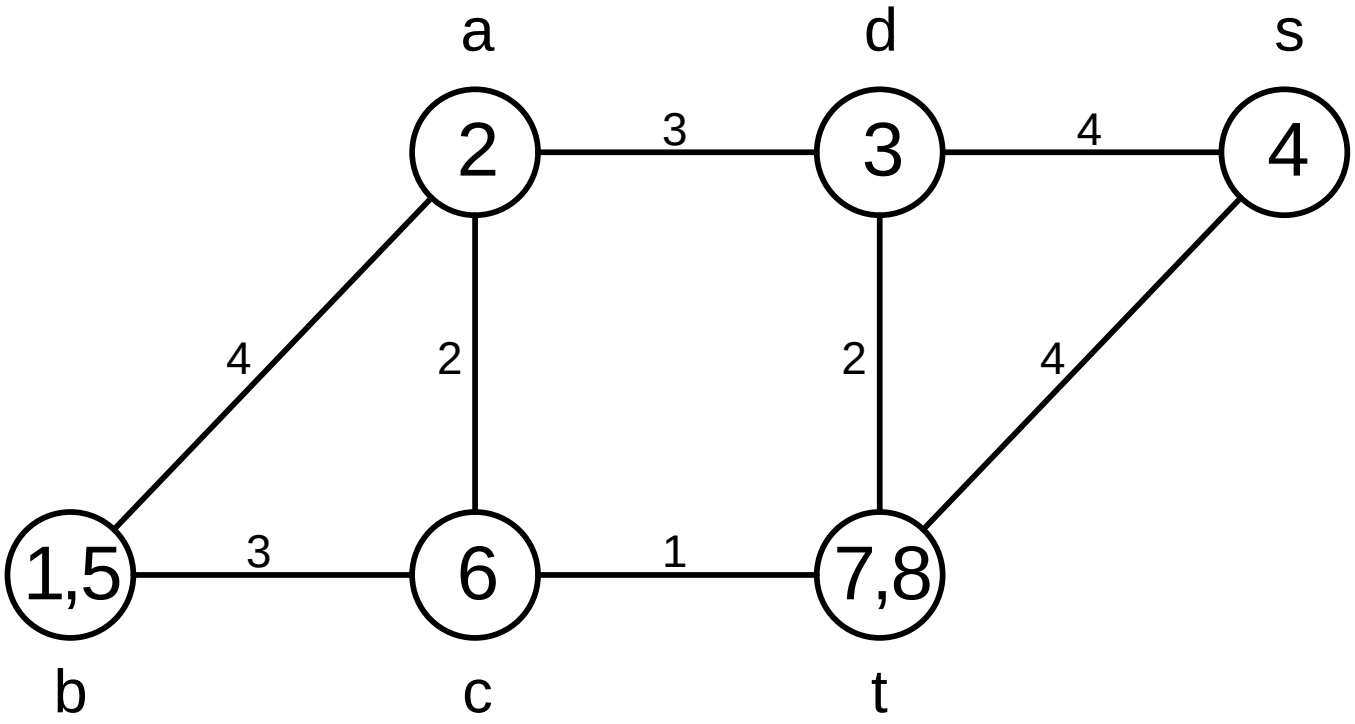
After the 2nd MinimumCutPhase(G, !, a), a = 2



vertex ordering: a, b, c, d, e, s, t

cut-of-the-phase: {8}, {1, 2, 3, 4, 5, 6, 7} ! = 5

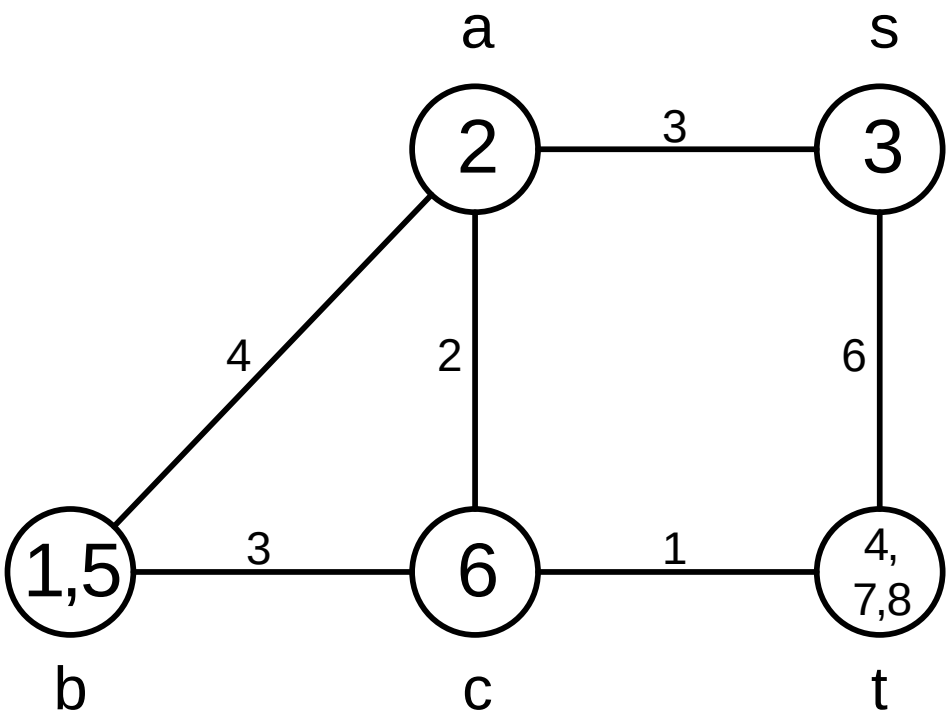
After the 3rd MinimumCutPhase(G, !, a), a = 2



vertex ordering: a, b, c, d, s, t

cut-of-the-phase: {7, 8}, {1, 2, 3, 4, 5, 6} ! = 7

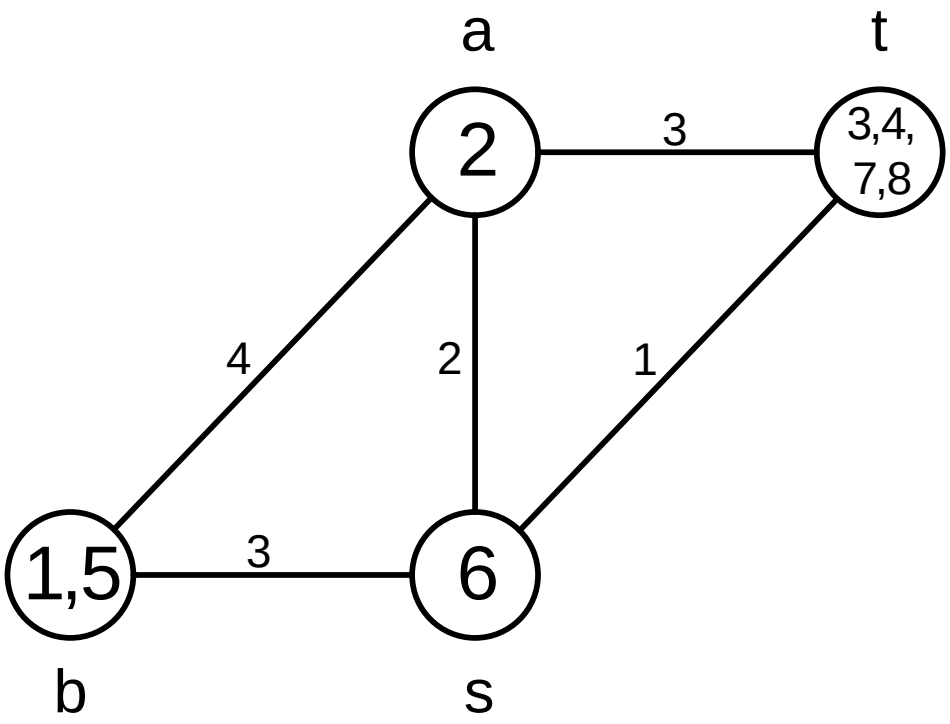
After the 4th MinimumCutPhase(G, !, a), a = 2



vertex ordering: a, b, c, s, t

cut-of-the-phase: {4, 7, 8}, {1, 2, 3, 5, 6} ! = 7

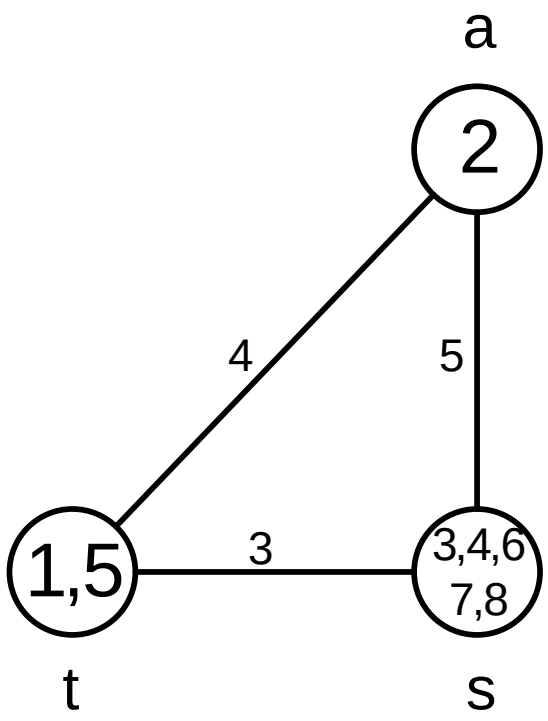
After the 5th MinimumCutPhase(G, !, a), a = 2



vertex ordering: a, b, s, t

cut-of-the-phase: {3, 4, 7, 8}, {1, 2, 5, 6} ! = 4

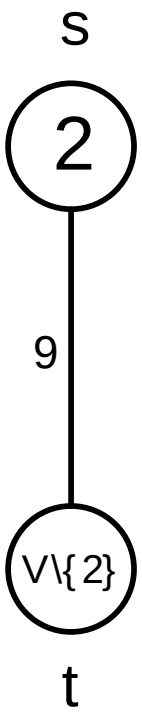
After the 6th MinimumCutPhase(G, !, a), a = 2



vertex ordering: a, s, t

cut-of-the-phase: $\{1, 5\}, \{2, 3, 4, 6, 7, 8\}$! = 7

After the 7th MinimumCutPhase(G, !, a), a = 2



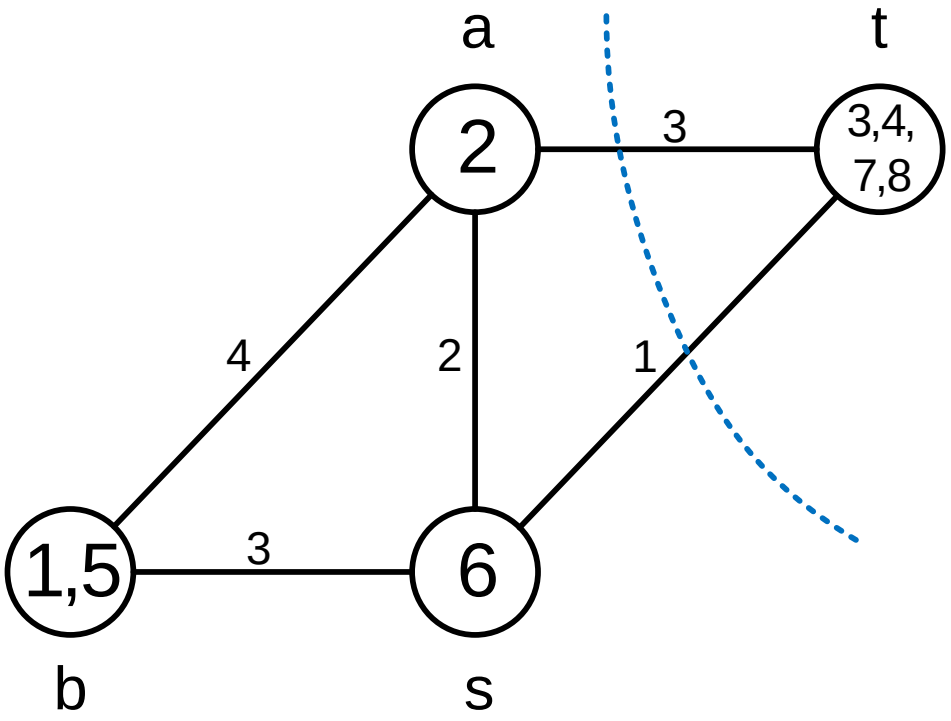
vertex ordering: s, t

cut-of-the-phase: {2}, {1, 3, 4, 5, 6, 7, 8} ! = 9

Cut-of-the-phase

cut-of-the-phase	!
$\{1\};\{2,3,4,5,6,7,8\}$	5
$\{8\};\{1,2,3,4,5,6,7\}$	5
$\{7,8\};\{1,2,3,4,5,6\}$	7
$\{4,7,8\};\{1,2,3,5,6\}$	7
$\{3,4,7,8\};\{1,2,5,6\}$	4
$\{1,5\};\{2,3,4,6,7,8\}$	7
$\{2\};\{1,3,4,5,6,7,8\}$	9

The Minimum Cut of the Graph G



vertex ordering: a, b, s, t

cut-of-the-phase: $\{3, 4, 7, 8\}, \{1, 2, 5, 6\}$ $! = 4$

Outline

Lemma

Each *cut-of-the-phase* is a minimum s-t cut in the current graph, where s and t are the two vertices added last in the phase.

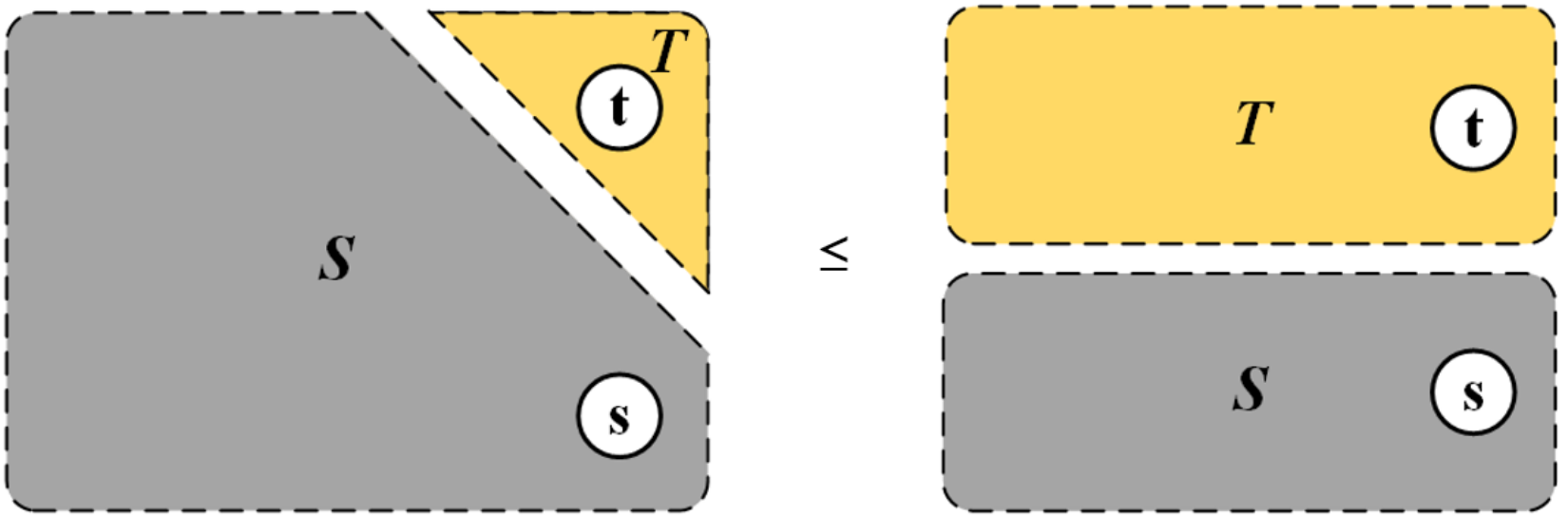
Theorem

The lightest of these *cuts-of-the-phase* is the minimum cut of G.

Assuming that the lemma holds, the theorem can be proved by a simple case distinction. Thus our proof is focused on the claimed property of the *cut-of-the-phase*.

Outline

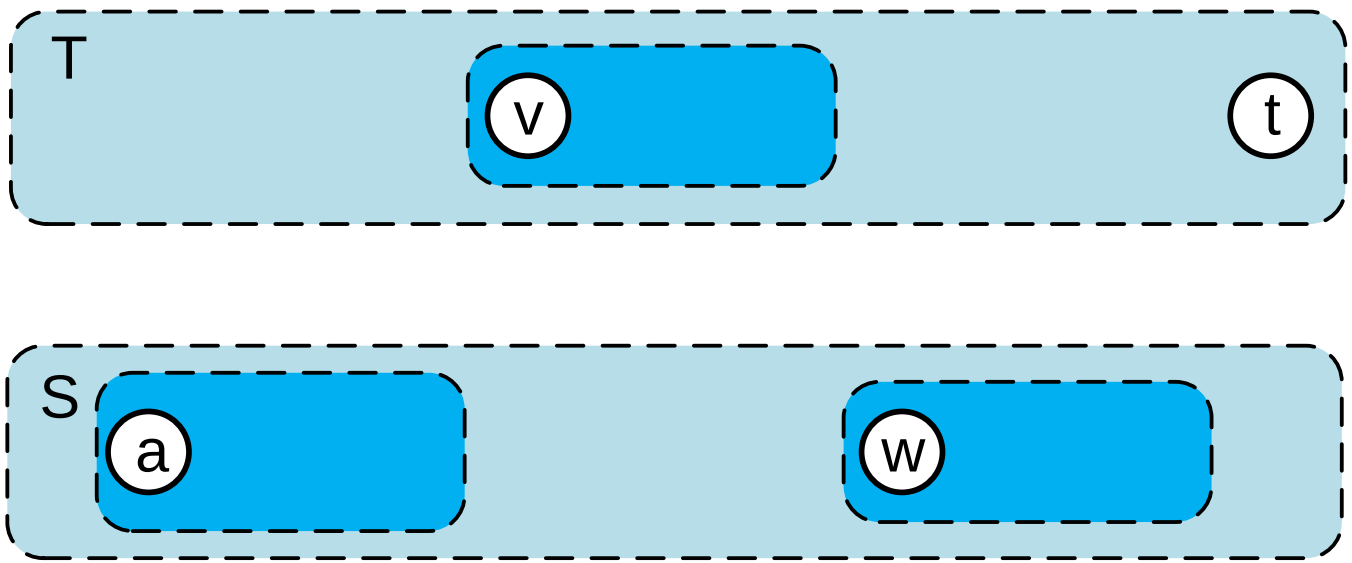
Proof



$$\omega(A_t, t) \leq \omega(C_t)$$

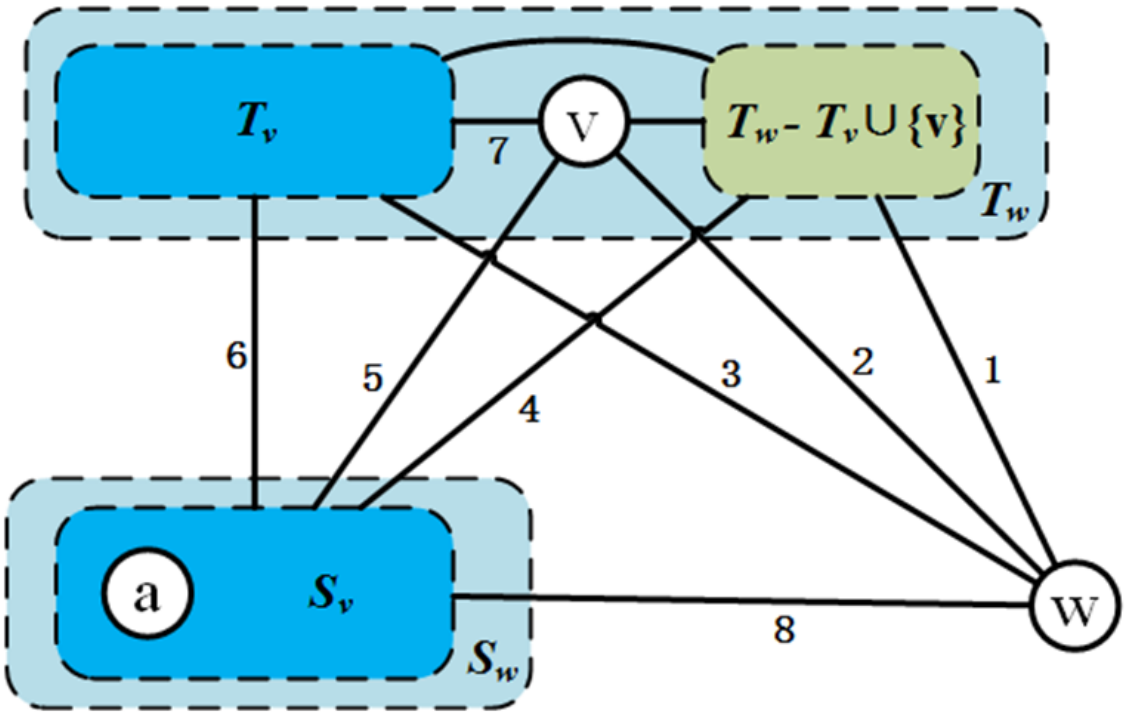
DeÞnition

We call a vertex v **active** when v and the vertex added just before v are in different parts of the cut.



active vertices: a, . . . , v, w, . . . , t

Proof

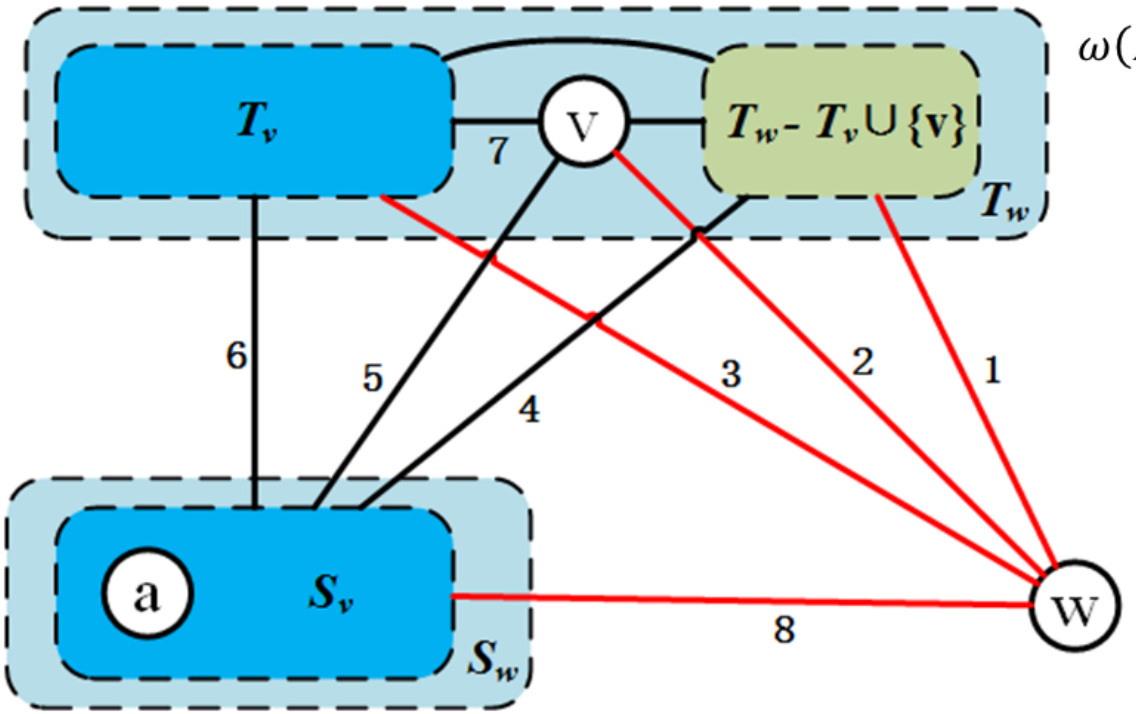


$$A_v = S_v \cup T_v$$

$$A_w = S_w \cup T_w$$

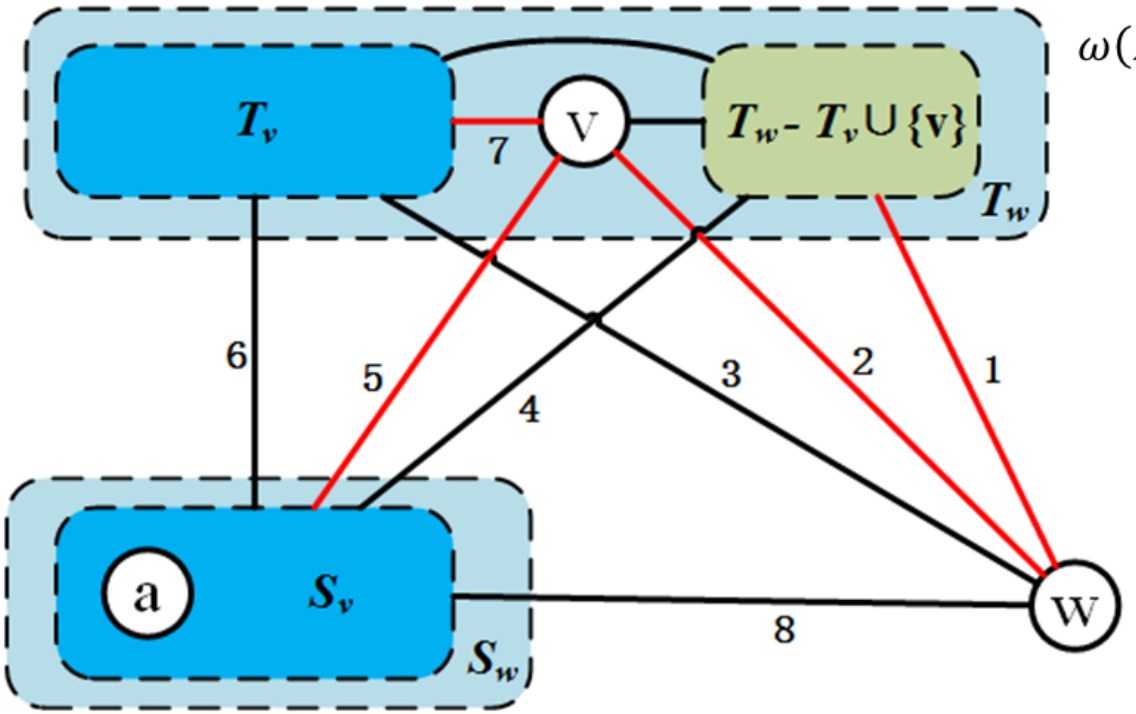
$$S_w = S_v$$

Proof



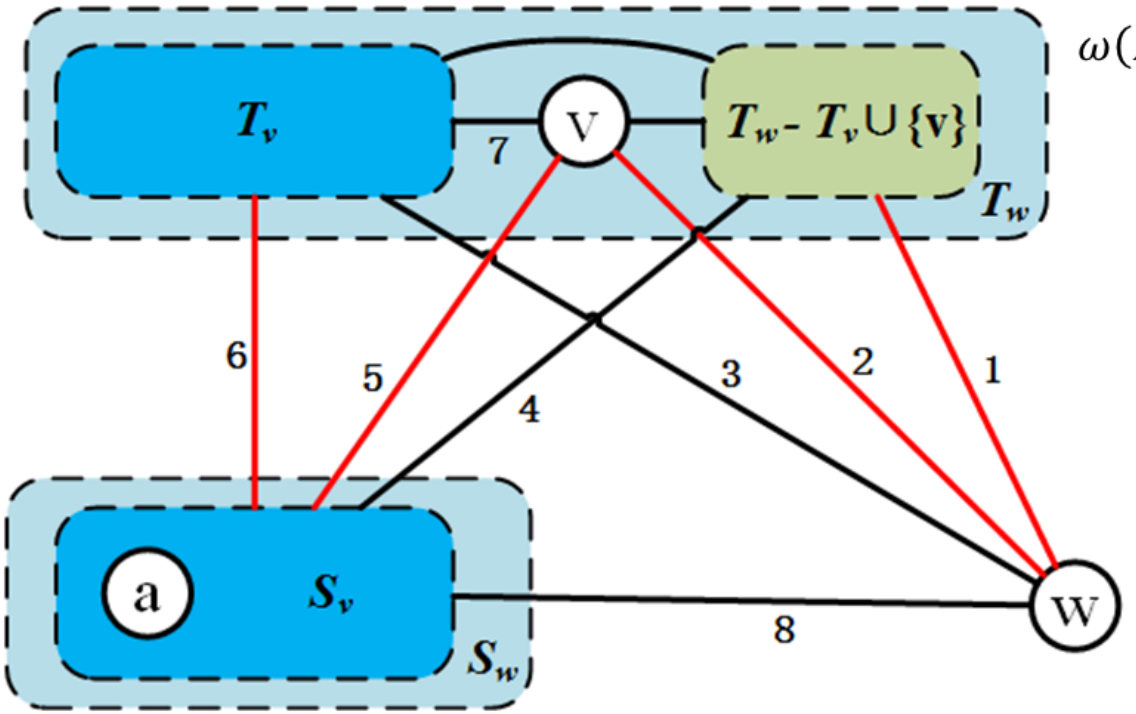
$$\omega(A_w, w) = (1 + 2) + (3 + 8)$$
$$= \omega(A_w \backslash A_v, w) + \omega(A_v, w)$$

Proof



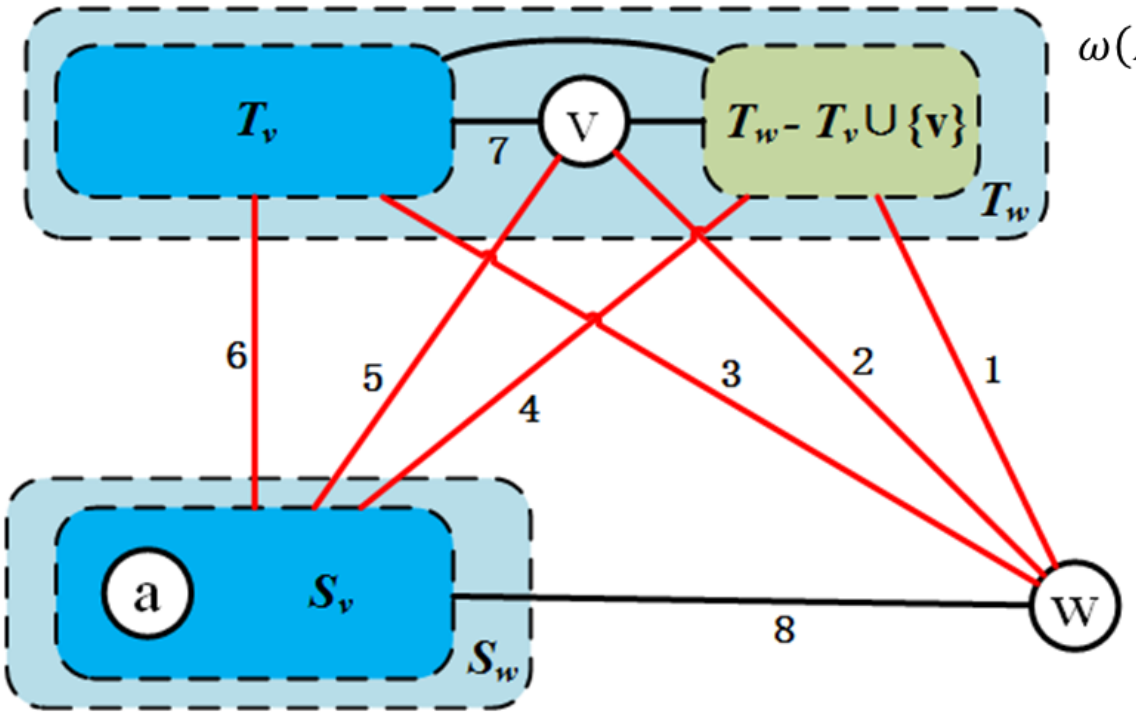
$$\omega(A_w, w) = (1+2) + (3+8)$$
$$= \omega(A_w \backslash A_v, w) + \omega(A_v, w)$$
$$\leq \omega(A_w \backslash A_v, w) + \omega(A_v, v)$$
$$= (1+2) + (5+7)$$

Proof



$$\omega(A_w, w) = (1+2) + (3+8)$$
$$= \omega(A_w \backslash A_v, w) + \omega(A_v, w)$$
$$\leq \omega(A_w \backslash A_v, w) + \omega(A_v, v)$$
$$= (1+2) + (5+7)$$
$$\leq \omega(A_w \backslash A_v, w) + \omega(C_v)$$
$$= (1+2) + (5+6)$$

Proof



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$$= (1+2)+(5+7)$$
$$\leq \omega(A_w \backslash A_v, w) + \omega(C_v)$$
$$= (1+2)+(5+6)$$
$$\leq \omega(C_w)$$
$$= (1+2+3+4+5+6)$$

Outline

MinimumCut Algorithm

```
MinimumCut (G, w, a)
1 while (| V |> 1) do
2 |   MinimumCutPhase(G, w, a)
3 |   if the cut-of-the-phase is lighter than the current minimum cut
  |   then
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```

MinimumCutPhase Algorithm

MinimumCutPhase(G, w, a)

```
1 A ! {  a}
2 while (A ≠ V) do
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MinimumCutPhase Algorithm

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Proposition

The running time for a single phase is $O(|E| + |V| \log |V|)$.

Proof.

1. All vertices that are not in A reside in a priority queue.

□

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2. Whenever a vertex x is added to A,
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2. Whenever a vertex x is added to A,
   the priority queue does ExtractMax and IncreaseKey operations.
3. $|V|$ ExtractMax $|E|$ IncreaseKey
4. Fibonacci heaps
   ExtractMax $O(\log |V|)$
   IncreaseKey $O(1)$
   Time for one phase is $O(|E| + |V| \log |V|)$.

Proposition

The overall running time of Stoer-Wagner algorithm for minimum cut in undirected graphs is $O(|V||E| + |V|^2 \log |V|)$.

Proof.

1. $|V|$ # 1 phases
2. Time for single phase $O(|E| + |V| \log |V|)$
3. Overall running time $O(|V||E| + |V|^2 \log |V|)$

□

Thank you!

Q & A